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Non-Linear Bending of a Circular Plate under Normal Pressure

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1. Introduction

The purpose of this paper is to determine the stresses and deformations produced by a uniform normal load on a thin circular plate for a wide range of values of the load and the thickness of the plate. Three types of support at the boundary are considered and in every case edge effects occur when the load is large and the plate is very thin. As is generally the case, these effects are accompanied by difficulties in obtaining numerical solutions by customary methods and special mathematical treatment becomes necessary. A method based on boundary layer theory has been developed and is presented here which makes it possible to get around these difficulties and obtain satisfactory approximations to the solution.

We take as our point of departure the pair of non-linear partial differential equations originally derived by v. Kármán [6] for the case of plates with large deflections. Since we confine our interest to solutions possessing radial symmetry[1], we can reduce these relations to ordinary differential equations which can be solved numerically by any one of several well-known methods as long as the solution does not vary too rapidly near the boundary.

A compact form of these equations is derived in Section 2. These expressions show that for each mode of support the character of the solution depends on a single parameter k, which is proportional to $(N/E)^{1/3} \cdot (R/h)^{4/3}$, where N is the normal load per unit area, E the modulus of elasticity, R the radius of the plate and h its thickness. For values of $k \leqq 1$, satisfactory solutions can be obtained by a perturbation expansion with respect to k^2 and this is done in Section 3. For larger values, say, $1 \leqq k \leqq 15$, a power series in $(r/R)^2$ is used in Section 4.

However, if we attempt to go to still larger values of k (large load, very

[1]This assumption is not always warranted. Thus, for example, wrinkling may occur for a sufficiently large load. This question has been investigated by Yanowitch [9], who used some of the results obtained in this paper to show that the radially symmetric solution of the general problem is unstable under certain conditions.

thin plate) the computations become excessively laborious. These difficulties, together with other characteristics of the solutions as k increases, form a pattern which is common to systems with boundary layers and which is described in detail in Section 5. Physically it is typical that the bending stresses become negligible compared to the membrane stresses over most of the plate but change rapidly and reach significant values in a narrow strip near the edge; mathematically, the development of a boundary layer is characterized by the fact that the coefficient of the highest order derivative in one of the equations goes to zero as k approaches infinity.

Boundary layer theory has been applied to similar problems of the deformation of plates in the past by Friedrichs and Stoker [3, 4], Chien [2], and Bromberg and Stoker [1]. In order to get solutions of sufficient accuracy in the present problem for values of k ranging from about 15 to infinity, it was found necessary to develop their asymptotic methods further. The method which is presented in Section 5 follows rather closely the approach described by Chien. It has the advantage that it permits approximations which are originally obtained separately for the boundary layer and the interior to be combined in a single expression having greater accuracy than either of its components.

In a considerably earlier investigation, Hencky [5] obtained a general solution to the problem of a normally loaded circular plate with pinned edges in which he made the additional assumption that the plate was so thin that its resistance to bending could be neglected. The effect of such an assumption is to reduce the order of the system of differential equations and consequently also the number of boundary conditions which can be satisfied. This simplification does not indicate which boundary conditions should be retained; thus Hencky was forced to make a conjecture about the appropriate reduced system of boundary conditions. The Hencky solution turns out to be the limit of the general solution in the interior of the plate as k goes to infinity. In Section 5 Hencky's assumption about the boundary conditions is verified and a procedure is indicated for determining the appropriate choice of boundary conditions in similar problems.

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2. Mathematical Formulation

The study of large deflections of plates requires that one of the basic assumptions of the linear theory of bending be rejected; that is, the normal displacements can no longer be considered small in comparison with the thickness of the plate. By making, instead, the assumption that the squares of the slopes of the middle surface were comparable to its strains. v. Kármán was led [6, 7] to the following pair of non-linear partial differential equations for the stresses and displacements:

$$(2.1)\qquad (\bar{\gamma}h)^2 E\nabla^4 w = \frac{N}{h} + \frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y},$$

$$(2.2)\qquad \nabla^4\Phi = E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]$$

where

$$(2.3)\qquad \nabla^4 = \nabla^2\cdot\nabla^2;\qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The quantity w is the deflection of the middle surface, N is the normal load per unit area, h is the thickness, E is the modulus of elasticity, $\bar{\gamma}^2 = \frac{1}{12(1-\nu^2)}$, ν is Poisson's ratio, and Φ is a stress function which is related to the stresses in the middle surface by the relations

$$(2.4)\qquad \sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2},\qquad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2},\qquad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}.$$

Since we assume that the plate will deform with radial symmetry, we may treat all dependent variables as functions only of the distance from the

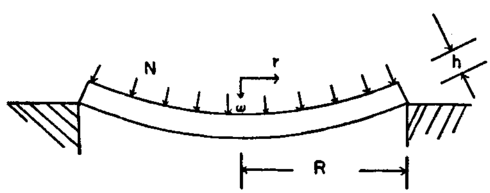

Figure 1
Elastic Plate in Buckled Condition

center, $r = (x^2+y^2)^{1/2}$ (see Figure 1). Equations (2.1) and (2.2) can now be written as

$$(2.5)\qquad \bar{\gamma}^2 h^2 E\nabla^4 w = \frac{N}{h} + \frac{1}{r}\left(\frac{dw}{dr}\frac{d^2\Phi}{dr^2} + \frac{d^2 w}{dr^2}\frac{d\Phi}{dr}\right),$$

$$(2.6)\qquad \frac{1}{E}\nabla^4\Phi = -\frac{1}{r}\frac{dw}{dr}\frac{d^2 w}{dr^2}$$

and the operator ∇^2 becomes

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{2.7}$$

The radial and circumferential stresses σ_r and σ_c are related to the stress function by the expressions

$$\sigma_r = \frac{1}{r}\frac{d\Phi}{dr}, \qquad \sigma_c = \frac{d^2\Phi}{dr^2}. \tag{2.8}$$

The bending moment μ_r and the radial bending stress σ_b can be expressed as functions of the displacement:

$$\begin{aligned} \mu_r &= -\bar{\gamma}^2 h^3 E\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right), \\ \sigma_b &= -6\bar{\gamma}^2 h E\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right). \end{aligned} \tag{2.9}$$

Deferring for the moment the statement of the boundary conditions at $r = R$, equations (2.5) and (2.6) can each be integrated once, using the conditions of regularity at $r = 0$ to fix the constants of integration. We thus obtain

$$\bar{\gamma}^2 h^2 E r \frac{d}{dr}(\nabla^2 w) = \frac{pr^2}{2h} + \frac{dw}{dr}\frac{d\Phi}{dr}, \tag{2.10}$$

$$\frac{r}{E}\frac{d}{dr}(\nabla^2\Phi) = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2. \tag{2.11}$$

These equations can be simplified by introducing new dimensionless variables

$$z = r^2 R^{-2}, \tag{2.12}$$

$$q = -\left(\frac{Eh}{NR}\right)^{1/3}\frac{R}{r}\frac{dw}{dr}, \tag{2.13}$$

$$p = -\left(\frac{Eh}{NR}\right)^{2/3}\frac{1}{E}\frac{1}{r}\frac{d\Phi}{dr}, \tag{2.14}$$

and a parameter k:

$$k = \frac{R}{\bar{\gamma}h}\left(\frac{NR}{Eh}\right)^{1/3}. \tag{2.15}$$

The variable p is proportional to the radial stress but it does not obey the customary sign convention; instead, positive values correspond to compressions. The variable q is proportional to dw/dz and will be referred to frequently as the slope; near the edge q is roughly proportional to the physical slope dw/dr. Dimensionless variables p_c and m_r, proportional to the circumferential stress and the bending moment, respectively, can be defined in terms of the new variables

$$p_c = p + 2zp',$$

(2.16)

$$m_r = 2zq' + (1+\nu)q = \frac{k^2}{NR^2}\mu_r.$$

Primes will henceforth be used to indicate differentiation with respect to z.

Introducing the new quantities in (2.10) and (2.11), we obtain,

(2.17) $$8k^{-2}(zq)'' + 1 + 2pq = 0,$$

(2.18) $$8(zp)'' - q^2 = 0$$

and these together with the boundary conditions which we now proceed to formulate provide the basis for our investigations.

As stated in the introduction, three types of support at the circumference of the plate are considered. These can be described as follows: *Case* A. Pinned and clamped edge, with the circumferential strain and the slope equal to zero:

(2.19A)

$$p_c(1) - \nu p(1) = 2p'(1) + [1-\nu]p(1) = 0,$$
$$q(1) = 0.$$

Case B. Pinned edge free to rotate, with the circumferential strain and the bending moment equal to zero:

(2.19B)

$$p_c(1) - \nu p(1) = 2p'(1) + [1-\nu]p(1) = 0,$$
$$m_r(1) = 2q'(1) + [1+\nu]q(1) = 0.$$

Case C. Simply supported edge with the radial stress and the radial bending moment both equal to zero:

(2.19C)

$$p(1) = 0,$$
$$m_r(1) = 2q'(1) + [1+\nu]q(1) = 0.$$

In addition, we assume in every case that the solutions are finite at the center of the plate:

(2.20) $$p(0),\quad q(0) \quad \text{finite}.$$

Equations (2.17) through (2.20) constitute a complete statement of the problem. It can be seen that any particular choice of the parameters of the problem as formulated originally reduces to the fixing of only two parameters, ν and k. Of these, ν has in practice a narrow range of variation which does not affect the solution qualitatively; in all numerical calculations we shall set it equal to 0.3. There remains, therefore, only the parameter k, and we proceed to seek solutions for its entire range of values, from zero to infinity, using the methods mentioned in Section 1.

3. Perturbation Method

For small values of the parameter it is natural to use a perturbation method; that is, to develop p and q as expansions in k^2:

$$\begin{aligned} p(z) &= \pi_0(z)+k^2\pi_1(z)+k^4\pi_2(z)+\dots, \\ q(z) &= \chi_0(z)+k^2\chi_1(z)+k^4\chi_2(z)+\cdots \end{aligned} \tag{3.1}$$

which are then substituted into equations (2.17) and (2.18) and the boundary conditions (2.19) and (2.20). The resulting expressions must be satisfied for each power of k^2 separately. This requirement yields a sequence of equations in π_i and χ_i which can be integrated directly if taken in order of increasing subscripts.

It can readily be determined that π_0, χ_0 and π_1 are identically zero for all three cases. The first non-zero terms in the expansions for p and q are *Case* A: pinned and clamped edge

$$\begin{aligned} \chi_1 &= .0625-.0625z, \\ \pi_2 &= (-.238+.244z-.16z^2+.041z^3)\times 10^{-3}. \end{aligned} \tag{3.2A}$$

Case B: pinned edge free to rotate,

$$\begin{aligned} \chi_1 &= .159-.0625z, \\ \pi_2 &= (-3.68+1.57z-.41z^2+.041z^3)\times 10^{-3}. \end{aligned} \tag{3.2B}$$

Case C: simply supported edge,

$$\begin{aligned} \chi_1 &= .159-.0625z, \\ \pi_2 &= (-1.201+1.573z-.413z^2+.041z^3)\times 10^{-3}. \end{aligned} \tag{3.2C}$$

The solutions χ_1 possess no novelty since they are in every case precisely those which are obtained from the linear theory of bending when the membrane stresses are neglected. This is not surprising since the assumption that k, and hence the normal load, are small leads naturally to the linearized theory.

The quantities π_2 correspond to the membrane stresses which are produced by the deformation. These are generally disregarded in linear theory because they are so small compared to the bending stresses.

The expression for the dimensionless circumferential stress in Case C is

$$p_c = k^4(-1.201+4.719z-2.065z^2+.287z^3)\times 10^{-3} \tag{3.3}$$

and we note that it changes from a negative value—a tension—in the central region to a positive value, corresponding to a compression, near the edge. This is of some interest because it indicates that one of the preconditions for instability and wrinkling around the edge—namely a band under compressive stresses which increase with the load—begins to manifest itself under conditions of very small loading. On the other hand, the circumferential stresses do not become compressive anywhere in the plate in Cases A or B.

The next non-zero terms in the expansions of (3.1) are χ_3 and π_4. These can easily be computed by the methods indicated at the beginning of this section. However, they are of little interest since when k is less than unity their contributions to q and p, respectively, are small and when k is greater than unity the power series method described in the next section is more useful than the perturbation method.

4. Power Series Method

This method is particularly useful for values of k greater than unity and was used by Way [8] and Stoker and Friedrichs. The variables p and q are replaced in the equations (2.17) and (2.18) and the boundary conditions (2.19) and (2.20) by power series in the variable z:

$$\begin{aligned} p &= \sum_{n=0}^{\infty} \beta_n z^n, \\ q &= \sum_{n=0}^{\infty} \varkappa_n z^n, \end{aligned} \tag{4.1}$$

where the β_n and $\varkappa_n$ are constant coefficients. Since the resulting expressions must be satisfied independently for each power of z, we obtain the following set of algebraic relations among the β_n, $\varkappa_n$:

$$\begin{aligned} \varkappa_1 &= -\frac{k^2}{8}\left(\tfrac{1}{2}+\beta_0\varkappa_0\right), \\ \varkappa_n &= -\frac{k^2}{4n(n+1)}\sum_{j=0}^{n-1}\beta_j\varkappa_{n-j-1}, \qquad n = 2, 3, 4, \cdots, \\ \beta_n &= \frac{1}{8n(n+1)}\sum_{j=0}^{n-1}\varkappa_j\varkappa_{n-j-1}, \qquad n = 1, 2, 3, \cdots, \\ &\beta_0,\ \varkappa_0 \text{ finite,} \end{aligned} \tag{4.2}$$

together with the various boundary conditions

$$\text{(4.3)}\qquad \begin{aligned} &\textit{Case A:} \quad \sum_{n=0}^{\infty} (2n+1-\nu)\beta_n = 0, \qquad \sum_{n=0}^{\infty} \varkappa_n = 0, \\ &\textit{Case B:} \quad \sum_{n=0}^{\infty} (2n+1-\nu)\beta_n = 0, \qquad \sum_{n=0}^{\infty} (2n+1+\nu)\varkappa_n = 0, \\ &\textit{Case C:} \quad \sum_{n=0}^{\infty} \beta_n = 0, \qquad \sum_{n=0}^{\infty} (2n+1+\nu)\varkappa_n = 0. \end{aligned}$$

Equations (4.2) and (4.3) constitute a complicated transcendental system which must be solved separately for each value of k by a cumbersome iterative technique. Our procedure consisted in making estimates of the values of β_0 and $\varkappa_0$ for each value of k, computing the β_n and $\varkappa_n$ by means of (4.2) and substituting these values in the summations in (4.3). The resulting deviations of the sums from zero were then used by the Newton-Raphson method to obtain improved estimates of β_0, $\varkappa_0$. This process was repeated until the values of β_0 and $\varkappa_0$ were determined to better than one percent.

The amount of labor involved in our procedure depends on the number of terms required to obtain accurate estimates of the summations in (4.3) as well as on the accuracy of the initial estimates of β_0 and $\varkappa_0$. It was found that the convergence characteristics of the series grew steadily worse as k increased and we did not obtain any solutions beyond $k = 14.1$. Particular difficulty was encountered in carrying out the computations for Case C.

Solutions obtained by the above methods for several different values of k are shown in Figures 2 through 9. These include graphs of p and q, the dimensionless radial stress and the slope, for all three cases, and of p_c and m_r, corresponding to the circumferential stress and the bending moment, for Case C alone. Every figure also contains a curve representing the limiting form of the solution in the interior, $0 \leqq z < 1$, as $k \to \infty$, which was obtained by methods described in the next section.

It should be noted in the graphs of the slope q that the region of maximum curvature moves closer to the edge as k increases and also that the curvature and the slope in this region are increasing. In view of the relationship of the bending moment m_r to the slope q as expressed in (2.16), this means that as k increases a region of rapid variation in the bending moment begins to appear near the edge. Figure 9 demonstrates this trend in m_r, which is particularly marked in Case C. This phenomenon corresponds to the edge effect which was referred to in the introductory section and which will be discussed in detail in the next section.

The graph of the slope for Case C, Figure 7, shows why this case presented the greatest computational difficulties: the variable q possesses a singularity at $z = 1$ in the limit as $k \to \infty$. It will be shown in the next section that accurate solutions can be obtained near the edge in this case by introducing new variables in place of p and q.

The graph of the variable p_c for Case C, Figure 8, shows that the circumferential stress continues to be tensile in the central region and compressive in an outer strip of the plate, as we found in the previous section for small values of k. As before, also, the circumferential stresses in Cases A and B are tensile throughout the plate.

In comparing the solutions for $k = 14.1$ and for the limit in the interior as $k \to \infty$, we find that the dimensionless radial stresses for $k = 14.1$ do not differ markedly from the limit solutions. On the other hand, there are considerable and significant differences in the slopes, and hence in the bending moments. There is also a significant difference in the circumferential stresses in Case C; it is much less marked, however, in Cases A and B, which are not shown. These differences indicate the desirability of obtaining solutions for large values of k and of considering the character of the solutions as $k \to \infty$.

5. Asymptotic Development

I. *General Discussion.* The object of this section is to obtain asymptotic developments for p and q as k becomes very large. From the results of the preceding section certain trends in the solutions can be noted. Most significant for our purposes is the following: As k increases (corresponding to increasing load N or decreasing relative thickness h/R) the slope q displays an increasingly marked difference between its character over a narrow ring bordering the edge which we call the boundary layer and that over an inner region which covers most of the plate and which we call the interior. The solution over the interior shows a rather smooth variation whereas the solution in the boundary layer is characterized by rapid change. The concept of boundary layers was originally introduced by Prandtl in studying the rapid change in speed near the boundary of a fluid flowing past a rigid wall.

If the solution over the interior for large k is extended smoothly to the boundary, the boundary conditions on q will not be satisfied in any of the three cases under study. The boundary layer can, therefore, be described as a region of transition in which the interior solution is modified to satisfy the actual boundary conditions.

Both the bending moment and the radial bending stress as well as q follow this pattern. Furthermore, for k large, they are very small in the interior and considerably larger in the boundary layer. This corresponds to the physical interpretation that the resistance of the plate to bending becomes negligible as k grows large and the bending stresses become negligible compared to the "membrane" stresses everywhere except in the boundary layer where the large curvature causes the bending stresses to be significant. This is also referred to as the edge effect.

The effects just described indicate that the solution for p and q in the limit as $k \to \infty$ may not be continuous and hence that the solutions of the fundamental system of equations (2.17) and (2.18) for k finite may not converge uniformly to a limit as $k \to \infty$. This can be deduced analytically from the fact that the coefficient of the highest order derivative in (2.17) is k^{-2}, so that this term drops out as $k \to \infty$. The order of the system of equations is thus reduced in the limit from four to two. A continuous solution of the reduced system of equations cannot in general satisfy all four boundary conditions. Since solutions of the original system for finite values of k do satisfy all four conditions, the limit of these solutions must be discontinuous.

In our problems, the limit solution is continuous over any subinterval $0 \leqq z \leqq \delta < 1$ (the interior) and satisfies the reduced system over the semi-closed interval $0 \leqq z < 1$. This is referred to as the interior solution. The limit solution is discontinuous at $z = 1$. The form of the general solution in the neighborhood of the edge for k very large is called the boundary layer solution.

A knowledge of the boundary conditions which the interior solution satisfies at the edge would permit us to obtain the interior solution directly from the reduced system of equations. However, the foregoing considerations do not indicate how the proper conditions can be determined analytically and one usually resorts to conjectures based on physical intuition or on trends of numerical results obtained for finite values of k.

The fact that sequences of solutions for increasing k in the three cases under consideration do not converge uniformly indicates that the general expression for a solution, which shall be called $g(z, k)$, is not an analytic function of k^{-1}. It is found, rather, that $g(z, k)$ has an essential singularity at $k^{-1} = 0$. The typical solution $g(z, k)$, therefore, cannot be expanded as a power series in k^{-1} as is done in the perturbation method.

In order to investigate the nature of these solutions, especially near the boundary, we introduce a new variable

$$x = 1 - z \tag{5.1}$$

and break g up into two parts (as was done by Chien), of which one contains the singularity and the other is regular at $k^{-1} = 0$; thus

$$g(z, k) = G(x, k) = \Gamma(x, \varepsilon) + \sum_{n=0}^{\infty} \varepsilon^n G_n(x) \tag{5.2}$$

where ε is some negative power of k. For solutions of Cases A and B, $\varepsilon = k^{-1}$

and the functions $G_n(x)$ are bounded and regular in the interval $0 \leqq x \leqq 1$. (We shall henceforth use upper case to represent functions of x and lower case those of z; thus $G(x) = g(z)$). Then the G_n can be expanded as a power series:

$$G_n(x) = \sum_{r=0}^{\infty} G_{nr} x^r = \sum_{s=0}^{\infty} g_{ns} z^s. \tag{5.3}$$

The two parts of G in (5.2) are so chosen that the asymptotic expansion of the function $\Gamma(x, k)$ with respect to k^{-1} for x positive has all its terms equal to zero. (This would, for example, be true of e^{-kx}.) This function may have non-zero terms in the asymptotic expansion *at* $x = 0$:

$$\Gamma(0, k) \sim \sum_{n=0}^{\infty} k^{-n} \Gamma_{n0} . \tag{5.4}$$

Consequently the function $G(x, k)$ expressed as an asymptotic expansion with respect to k^{-1}, yields

$$G(x, k) \sim \sum_{n=0}^{\infty} k^{-n} G_n(x) \qquad \text{for } x > 0, \tag{5.5a}$$

$$G(x, k) \sim \sum_{n=0}^{\infty} k^{-n} [G_n(0) + \Gamma_{n0}], \qquad \text{at } x = 0. \tag{5.5b}$$

Thus, in the limit as $k \to \infty$ the solution for $G(x, k)$ becomes discontinuous at $x = 0$, approaching $G_0(x)$ in the interior and $G_0(0) + \Gamma_{00}$ on the boundary. This agrees with the previous remarks about the character of the solution in the limit.

It can now be seen that if we try to solve for $G(x, k)$ by the method of perturbations with respect to k^{-1} in the normal way (as in Section 3) we can only obtain the asymptotic series valid in the interior; the contribution of $\Gamma(x, k)$ will be overlooked. We shall refer to G_n as the n-th order interior solution.

It is generally possible to introduce a new variable $y = y(x, k)$ which will transform $\Gamma(x, k)$ into a function $\gamma(y, k)$ which has a non-vanishing asymptotic expansion for all values of y, if we treat y and k in γ as independent quantities. In particular in Cases A and B, if we set

$$y = kx = k(1 - z), \tag{5.6}$$

γ can be expanded in powers of k^{-1}:

$$\gamma(y, k) = \sum_{n=0}^{\infty} k^{-n} \gamma_n(y). \tag{5.7}$$

The $\gamma_n(y)$ are all integral functions of finite order which approach zero as y approaches infinity. This implies that $y^a \gamma_n(y) \to 0$ as $y \to \infty$ for any value of a, and the same is true of the product of y^a with the derivative of γ_n to any order. By substituting for y in (5.7),

$$\Gamma(x, k) = \sum_{n=0}^{\infty} k^{-n}\gamma_n(kx) \tag{5.8}$$

and it can be seen that the character of the γ_n is consistent with the vanishing of all terms in the asymptotic expansion of Γ with respect to k^{-1} for positive values of x. From (5.4) it follows that

$$\gamma_n(0) = \Gamma_{n0}\,. \tag{5.9}$$

The function $g(z, k)$ can now be represented formally as a function of y and k:

$$\begin{aligned} g(z, k) = \bar{g}(y, k) &= \gamma(y, k) + \sum_{n=0}^{\infty} k^{-n}G_n(k^{-1}y) \\ &= \sum_{n=0}^{\infty} k^{-n}[\gamma_n(y) + \sum_{s=0}^{n} G_{n-s,s}y^s], \end{aligned} \tag{5.10}$$

and, therefore,

$$\bar{g}(y, k) = \sum_{n=0}^{\infty} k^{-n}\bar{g}_n(y), \tag{5.10$'$}$$

where $\bar{g}_n$ replaces the bracket in (5.10). It is instructive, in what follows, to express $g(z, k)$ also as a mixed function of x and y; namely,

$$\begin{aligned} g(z, k) = \gamma_0(y) &+ k^{-1}\gamma_1(y) + k^{-2}\gamma_2(y) + \cdots \\ +G_{00} &+ G_{01}x \quad + G_{02}x^2 \quad + \cdots \\ &+ k^{-1}G_{10} \;\; + k^{-1}G_{11}x + \cdots \\ &\qquad\qquad + k^{-2}G_{20} \quad + \cdots \\ &\qquad\qquad\qquad\qquad + \cdots. \end{aligned} \tag{5.11}$$

The reason for the particular ordering is that the rows correspond to $\gamma(y, k)$, $G_0(x)$, $G_1(x)$, etc. and the columns correspond to $\bar{g}_0(y)$, $\bar{g}_1(y)$, $\bar{g}_2(y)$, etc. The second and following rows correspond to the zero-th and higher order interior solutions; the columns are called the boundary layer solutions of zero-th and higher order.

Equations (5.10) and (5.11) demonstrate that as $y \to \infty$ for a fixed k the boundary layer solutions $\bar{g}_n(y)$ approach polynomials (since the $\gamma_n(y) \to 0$) and the coefficients in these polynomials are identical with those which appear in the power series for the interior solution. This correspondence takes the form

$$\left.\frac{d^rG_n}{dx^r}\right|_{x=0} = \left.\frac{d^r}{dy^r}(\bar{g}_{n+r} - \gamma_{n+r})\right|_{y=0}. \tag{5.12}$$

The procedure for obtaining interior, boundary layer and general solutions can now be formulated. The differential equations for the interior follow from the replacement of g by its asymptotic expansion (5.5a) in the interior or its equivalent in terms of z:

$$G(x, k) \sim \sum_{n=0}^{\infty} k^{-n} G_n(x) = \sum_{n=0}^{\infty} k^{-n} g_n(z), \tag{5.5a'}$$

and from the requirement that the expression be satisfied identically in k. The resulting sets of equations will each be of reduced order. The boundary conditions at the center, $z = 0$, for the original system can similarly be transformed to furnish conditions on the g_n. The outer boundary conditions at $z = 1$ cannot be used, as we have seen, and some of them must be replaced in order to get a completely formulated interior problem. In general these replacements are obtained in connection with the boundary layer problem.

The latter is derived by substituting the variable y of (5.6) for x or z in the original differential equations and by using the expansion (5.10′) for $\bar{g}(y, k)$ to obtain a sequence of sets of equations for $\bar{g}_n(y)$. The parameter ε in (5.2) was chosen equal to k^{-1} so that the sets of boundary layer equations are not reduced in order by this maneuver but retain the same order as the original one. Again, the question of boundary conditions arises. The conditions at $z = 1$ can be transformed into conditions on the $\bar{g}_n(y)$ at $y = 0$. However, we now lack conditions corresponding to those at $z = 0$. Thus, if we solve the incomplete interior and boundary layer problems, the solutions will have undetermined constants because of the absence of boundary conditions where the two regions "abut." These constants are determined by conditions obtained from the relationships (5.12). The latter express the consistency of the two solutions and are greater in number than are generally needed. This does not produce any inconsistencies since the redundant conditions are naturally satisfied. It is generally a simple matter to make a suitable selection from among them and so to evaluate the undetermined constants in both sets of solutions.

The two solutions can be combined to provide an approximation to the general solution for finite values of k which is better than either one taken alone. From (5.11) we see that successive interior solutions yield successive rows starting with the second; similarly the sequence of boundary layer solutions yields successive columns. Each set of n rows or n columns has terms of (5.11) which the other lacks and also terms which are duplicated. The combination

$$g_{an}(z, k) = \sum_{r=0}^{n} k^{-r}[g_r(z) + \gamma_r(y)] \tag{5.13}$$

furnishes the best n-th order approximation since it includes all terms in the g_n and $\bar{g}_n$ without any of the duplications.

The procedures just described must be modified in Case C to take account of the fact that the interior solutions possess singularities in x at $x = 0$. Such a singularity would seem, at first glance, to be unacceptable for physical reasons. However, as we have already seen, the asymptotic expansion of the complete solution at $x = 0$ includes terms other than those obtained from the interior solution; and these additional terms serve to cancel out the singularities and leave the complete solution regular. The singularities reflect the fact that the magnitude of the solutions near the boundary is proportional to some power of k.

The typical solution in Case C can be expressed in a form similar to that of (5.2),

$$G(x, k) = \Gamma(x, \varepsilon) + \sum_{n=0}^{\infty} \varepsilon^n G_n(x), \tag{5.14}$$

but with a different value of the parameter ε. The functions $G_n(x)$ all possess poles of order up to s at $x = 0$, so that they can be expanded in the form

$$G_n(x) = x^{-s} \sum_{r=0}^{\infty} G_{nr} x^r. \tag{5.15}$$

The asymptotic series for $G(x, k)$ in the interior,

$$G(x, k) \sim \sum_{n=0}^{\infty} \varepsilon^n G_n(x) \tag{5.16}$$

will therefore have a corresponding singularity at $x = 0$.

In order to study the boundary layer in this case we must revise the value of ε in

$$y = \varepsilon^{-1} x, \tag{5.17}$$

and the expansion of $\gamma(y, \varepsilon)$ becomes

$$\gamma(y, \varepsilon) = \varepsilon^{-s} y^{-s} \sum_{n=0}^{\infty} \varepsilon^n \gamma_n(y) \tag{5.18}$$

so that it may cancel out the poles in the G_n. The expressions (5.10) and (5.11) are modified accordingly:

$$\begin{aligned} \bar{g}(y, k) &= \varepsilon^{-s} y^{-s} \sum_{n=0}^{\infty} \varepsilon^n [\gamma_n(y) + \sum_{s=0}^{n} G_{n-s, s} y^s] \\ &= \varepsilon^{-s} y^{-s} \sum_{n=0}^{\infty} \varepsilon^n \bar{g}_n(y), \end{aligned} \tag{5.19}$$

$$\begin{aligned} g(z, k) = x^{-s}[\gamma_0(y) + \varepsilon\gamma_1(y) + &\cdots \\ + G_{00} \quad + G_{01}x \quad + &\cdots \\ + \varepsilon G_{10} \quad + &\cdots \\ + &\cdots]. \end{aligned} \tag{5.20}$$

Not only are the γ_n integral functions of finite order which go to zero as $y \to \infty$ in this case; they also have a form which causes $y^{-s}\bar{g}_n(y)$ to remain bounded as y approaches zero.

The procedure for obtaining interior and boundary layer solutions is in principle the same for Case C as for the other two cases. In the following paragraphs we apply the above methods to obtain solutions for the three problems described in Section 2.

II. *Case* A (pinned and clamped edges). As described above, a sequence of sets of differential equations is obtained for the interior problem by substituting the expansions

$$\sum_{n=0}^{\infty} k^{-n} p_n(z) \quad \text{and} \quad \sum_{n=0}^{\infty} k^{-n} q_n(z) \tag{5.21}$$

for p and q, respectively, in the original equations (2.17) and (2.18) and setting the resulting expressions equal to zero identically in k. This yields

$$\begin{aligned} 1+2p_0q_0 &= 0, \\ 8(zp_0)''-q_0^2 &= 0, \end{aligned} \tag{5.22}$$

$$\begin{aligned} p_0q_1+p_1q_0 &= 0, \\ 4(zp_1)''-q_0q_1 &= 0, \end{aligned} \tag{5.23}$$

$$\begin{aligned} 4(zq_{m-2})''+\sum_{j=0}^{m} p_j q_{m-j} &= 0, \\ 8(zp_m)''-\sum_{j=0}^{m} q_j q_{m-j} &= 0, \end{aligned} \qquad m = 2, 3, \cdots. \tag{5.24}$$

The boundary conditions for the p_m and q_m are: they remain finite at $z = 0$ and satisfy relations (5.12) at $z = 1$ when transformed into corresponding functions of x. It can be seen that the order of the system of equations has been reduced from four to two.

Solutions can be obtained successively for the p_n and q_n by the power series method. The results for p_0 and p_1 which satisfy the condition at $z = 0$ are

$$\begin{aligned} p_0 &= -\frac{a_0}{4} \sum_{n=0}^{\infty} p_{0n} \left(\frac{z}{a_0^3}\right)^n, \\ p_1 &= a_1 \sum_{n=0}^{\infty} p_{1n} \left(\frac{z}{a_0^3}\right)^n, \end{aligned} \tag{5.25}$$

where the first ten coefficients are given in Table I. The quantities a_0 and a_1 are to be determined by applying conditions (5.12). Expressions or values of q_0 and q_1 can readily be obtained from the algebraic relations in (5.22) and (5.23). These results also apply to the other two cases; only the values of a_0 and a_1 are different and depend on the conditions at the edge.

n	p_{0n}	p_{1n}
0	1	1
1	− 1	2
2	− .667	3.33
3	− .722	5.77
4	− .944	10.39
5	− 1.370	19.19
6	− 2.125	36.13
7	− 3.452	69.05
8	− 5.803	133.5
9	−10.01	260.4

TABLE I

The boundary layer equations for the functions $\bar{p}_n(y)$ and $\bar{q}_n(y)$ in the expansions

$$\bar{p}(y, k) = \sum_{n=0}^{\infty} k^{-n}\bar{p}_n(y), \qquad \bar{q}(y, k) = \sum_{n=0}^{\infty} k^{-n}\bar{q}_n(y) \tag{5.26}$$

have the form

$$8\bar{q}_0'' + 1 + 2\bar{p}_0\bar{q}_0 = 0, \qquad \bar{p}_0'' = 0, \tag{5.27}$$

$$4(\bar{q}_n - y\bar{q}_{n-1})'' + \sum_{j=0}^{n} \bar{p}_j\bar{q}_{n-j} = 0, \qquad 8(\bar{p}_n - y\bar{p}_{n-1})'' - \sum_{j=0}^{n} \bar{q}_j\bar{q}_{n-j-2} = 0, \qquad n = 1, 2, \cdots, \tag{5.28}$$

where $\bar{q}_{-1}$ and $\bar{q}_{-2}$ are to be taken equal to zero. The boundary conditions at $y = 0$ are

$$\bar{q}_n(0) = 0, \qquad 2\bar{p}_n'(0) - (1-\nu)\bar{p}_{n-1}(0) = 0, \qquad n = 0, 1, 2, \cdots. \tag{5.29}$$

and $\bar{p}_{-1}$ is to be taken equal to zero. The boundary conditions at $y = \infty$ are: the $\bar{p}_n$ and $\bar{q}_n$ are to approach polynomials consistent with the relations (5.12).

The first two sets of solutions of (5.27) and (5.28) which also satisfy (5.29) are

$$\bar{p}_0 = -4a^2, \qquad \bar{p}_1 = -2a^2(1-\nu)y + \bar{p}_{10}, \qquad \bar{q}_0 = \frac{1}{8a^2}(1 - e^{-ay}), \tag{5.30}$$

$$\bar{q}_1 = -\frac{1-\nu}{16a^2}y + \frac{\bar{p}_{10}}{32a^4} - \frac{\bar{p}_{10}}{32a^4}\left(e^{-ay} + \frac{1}{2a}ye^{-ay}\right) - \frac{ye^{-ay}}{64a^2}(5 + \nu - [3-\nu]ay),$$

in which the constants a and $\bar{p}_{10}$ are yet to be determined.

The conditions (5.12) necessary to relate the interior and boundary layer solutions can be stated in the following way so as to provide the greatest insight. We first use the solutions $\bar{p}_0$ and $\bar{p}_1$ to write the first few terms of $\bar{p}(y, k)$ in accordance with the pattern of (5.11); thus,

$$\begin{aligned} \bar{p}(y, k) = -4a^2 - \frac{1}{k}\,2a^2(1-\nu)y &+ \cdots \\ + \frac{1}{k}\bar{p}_{10} \qquad &+ \cdots \\ &+ \cdots . \end{aligned} \tag{5.31}$$

We now use the interior solutions $p_0(z)$ and $p_1(z)$ to obtain the corresponding expansions of $P(x, k)$:

$$\begin{aligned} P(x, k) = p_0(1) - xp_0'(1) &+ \cdots \\ + \frac{1}{k}p_1(1) &+ \cdots \\ &+ \cdots . \end{aligned} \tag{5.32}$$

The relations (5.12) indicate that

$$\begin{aligned} p_0(1) &= -4a^2, \\ p_0'(1) &= 2a^2(1-\nu). \end{aligned} \tag{5.33}$$

Since $p_0(1)$ and $p_0'(1)$ depend only on the parameter a_0 we can solve the two equations (5.33) for a and a_0. This yields

$$a_0 = 1.72, \qquad a = 0.2885 \tag{5.34}$$

and the zero order solutions are completely determined.

The constants a_1 and $\bar{p}_{10}$ can be determined by the same method after the second order solutions of the boundary layer problem are obtained, yielding

$$a_1 = 1.022, \qquad \bar{p}_{10} = 1.618, \tag{5.35}$$

and these complete the numerical solution of the first order problems. (These solutions have also been obtained by Chien.)

Since $\bar{p}_0(y)$ and $\bar{p}_1(y)$ turn out to be polynomials, they simply duplicate some of the terms in $P_0(x)$ and $P_1(x)$, as shown in (5.31) and (5.32). The same is not true of $\bar{q}_0$ and $\bar{q}_1$ compared with $Q_0(x)$ and $Q_1(x)$. The former contain terms which do not appear in the latter; namely, the exponential terms which make a significant contribution at $y = 0$ ($x = 0$ or $z = 1$) but which otherwise become small compared to the other terms in the neighborhood of $x = 0$ as y increases (i.e., $x = \delta > 0$ and k increasing). These terms are the integral functions of finite order previously referred to, and they give rise to the boundary layer phenomenon. No such transcendental terms are found in $\bar{p}_0$ and $\bar{p}_1$, but they appear in $\bar{p}_2$ and all higher order $\bar{p}_n$. It is customary to say that there is no boundary layer effect to zero or first order in the quantity p, but that there is such an effect in q.

It can be shown that, in general, p will not have zero-th and first order boundary layer effects if it has the form (5.11) rather than (5.20) and, if its value at $x = 0$ approaches a non-zero constant as $k \to \infty$. For, if $\bar{p}_0(y)$ and $\bar{p}_1(y)$ contained integral functions of finite order $\pi_0(y)$ and $\pi_1(y)$, respectively, the complete solution for $p(z, k)$ could be written as

$$\begin{aligned} p(z, k) &= \bar{p}(y, k) \\ &= \pi_0(y)+P_{00}+\frac{1}{k}[\pi_1(y)+P_{10}+P_{01}y]+\sum_{n=2}^{\infty}k^{-n}\bar{p}_n(y). \end{aligned} \tag{5.36}$$

The original equation (2.18) can be rewritten as

$$q^2 = 8zp''+16p'. \tag{2.18'}$$

Substituting (5.36) in this expression, it follows that

$$q^2 = k^2 \cdot 8z\pi_0''+k[-16\pi_0'+8z\pi_1'']+\text{additional terms}; \tag{5.37}$$

the "additional terms" are representable as a series in zero and negative powers of k multiplied by regular functions of y. From (5.37) it can be seen that if q is to remain bounded as k increases, then π_0'', π_0' and π_1'' must equal zero everywhere, and it follows that π_0 and π_1 are identically zero. These conclusions do not apply to higher order terms π_2, π_3, $\cdots$ since these appear only among the "additional terms" in (5.37), which remain bounded as k increases.

It is of interest to note the effect of the absence of zero-th and first order boundary layer effects in p on the boundary conditions in this quantity. In general, p can then be expressed as

$$p(z, k) = p_0(z)+\frac{1}{k}p_1(z)+\frac{1}{k^2}[\pi_2(y)+p_2(z)]+O\left(\frac{1}{k^3}\right). \tag{5.38}$$

Substituting this expression into the boundary condition

$$0 = 2p'(1)+(1-\nu)p(1), \tag{2.19A}$$

we obtain

$$0 = 2p_0'(1)+(1-\nu)p_0(1)+\frac{1}{k}[2p_1'(1)+(1-\nu)p_1(1)-\pi_2'(0)]+O\left(\frac{1}{k^2}\right). \tag{5.39}$$

This equation must be satisfied separately for each power of k. It therefore follows that the boundary condition for p_0 is the same as that for the function p itself. One may say that the interior problem (in zero-th order) "retains" the original boundary condition in p.

On the other hand, it can be seen that the interior expansion will not, in general, yield the correct boundary condition for p_1 and higher terms if second and higher order boundry layer effects exist. Similarly, the existence of a zero-th order boundary layer in q makes it impossible to apply the boundary conditions in q to the interior problem.

These results are noteworthy because they indicate that the interior problem can be completely formulated in zero-th order without reference to the boundary layer solutions in Cases A and B. They also explain why Hencky's solution of the problem of a loaded membrane with fixed circumference (neglecting bending) is correct. The differential equation for the loaded membrane is the same as in the zero-th order interior problem (5.22), and Hencky made the assumption that the boundary condition on p, in (2.19A), was retained. We can see from the foregoing discussion that this assumption is valid.

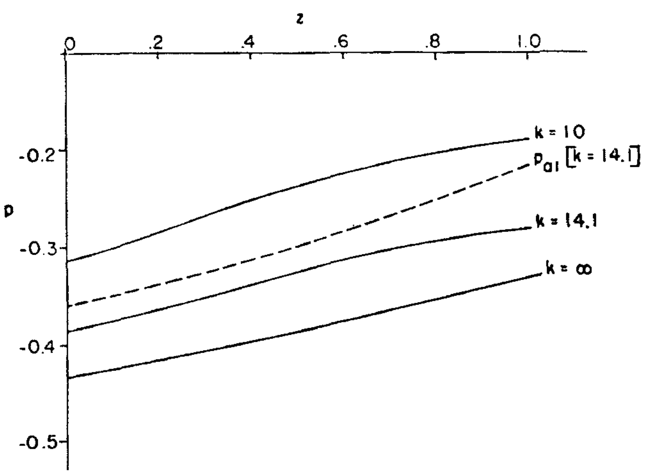

Figure 2
Dimensionless Radial Stress, Case A

The interior and boundary layer solutions can be combined as in equation (5.13); thus, if only the zero-th and first order solutions are included, we obtain

$$
\begin{aligned}
&p_{a1}(z, k) = p_0(z)+k^{-1}p_1(z), \\
(5.40)\qquad &q_{a1}(z, k) = \bar{q}_0(y)-\frac{1}{8a^2}+q_0(z)+k^{-1}\left[\bar{q}_1(y)+\frac{1-\nu}{16a^2}y-\frac{\bar{p}_{10}}{32a^4}+q_1(z)\right].
\end{aligned}
$$

These approximations are plotted in Figures 2 and 3 for $k = 14.1$ together with the power series solutions and the zero-th order interior solutions. It can be seen that q_{a1} provides a better approximation to the power series solution for q than the interior solution alone.

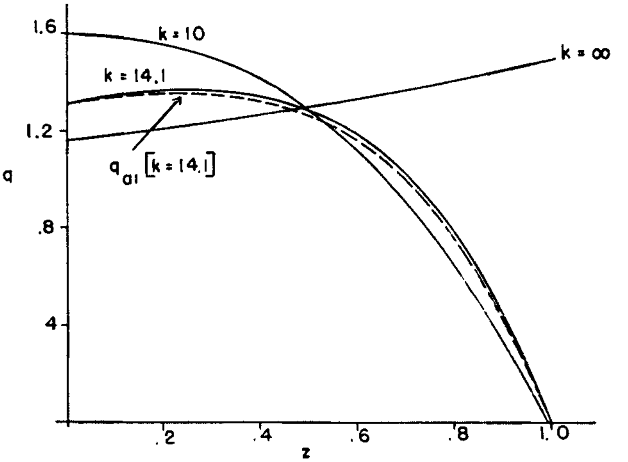

Figure 3
Dimensionless "Slope", Case A

III. *Case* B (pinned edges, free to rotate). The interior problem in this case is identical with that of Case A, and hence the solutions have the same form (5.25). The boundary layer equations are also the same as for Case A but the boundary conditions at $y = 0$ become

$$2\bar{p}'_n(0)-(1-\nu)\bar{p}_{n-1}(0) = 0, \quad 2\bar{q}_n(0)-(1+\nu)\bar{q}_{n-1}(0) = 0. \tag{5.41}$$

Using the same methods as for Case A, the boundary layer solutions turn out to be

$$\bar{p}_0 = -4a^2, \qquad \bar{q}_0 = \frac{1}{8a^2}, \qquad \bar{p}_1 = -2a^2(1-\nu)y, \qquad \bar{q}_1 = -\frac{1-\nu}{16a^2}y - \frac{1}{8a^3}e^{-ay}. \tag{5.42}$$

The zero-th order solution in p is the same as for Case A, so that $a = 0.288$ and $a_0 = 1.72$. The first order solution in p yields $a_1 = \bar{p}_{10} = 0$, that is, $p_1(z) \equiv 0$.

It can be seen that q does not have a "boundary layer effect" in the zero-th order in this case. The combined approximations can be obtained as in the preceding case and the results are plotted in Figures 4 and 5 which also provide a comparison with the power series solutions.

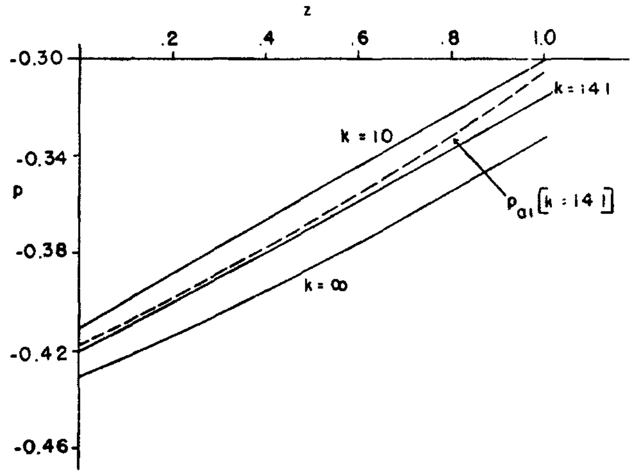

Figure 4
Dimensionless Radial Stress, Case B

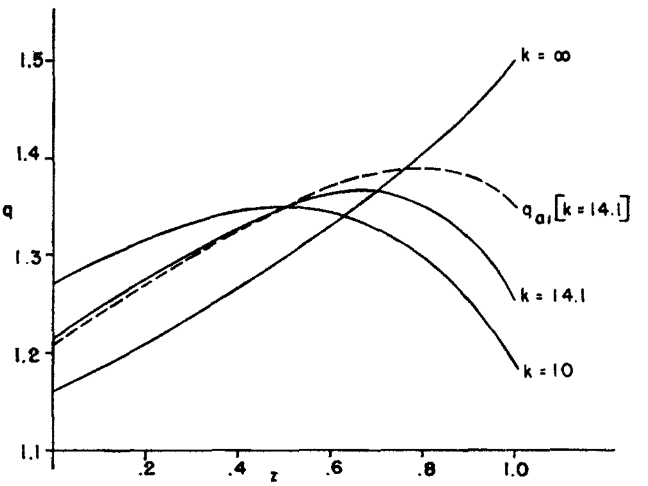

Figure 5
Dimensionless "Slope", Case B

IV. *Case* C (simple supported edges). The analytic difficulties which arise in attempting to solve this case are much greater than in the other two cases, and it was found expedient to obtain only the zero-th order approximation.

The interior solution for $p_0(z)$ is the same as in the preceding cases with a_0 again left undetermined. As before, it is possible to obtain a boundary condition for $p_0(1)$ without recourse to the first order solutions, but the reasoning is somewhat more involved:

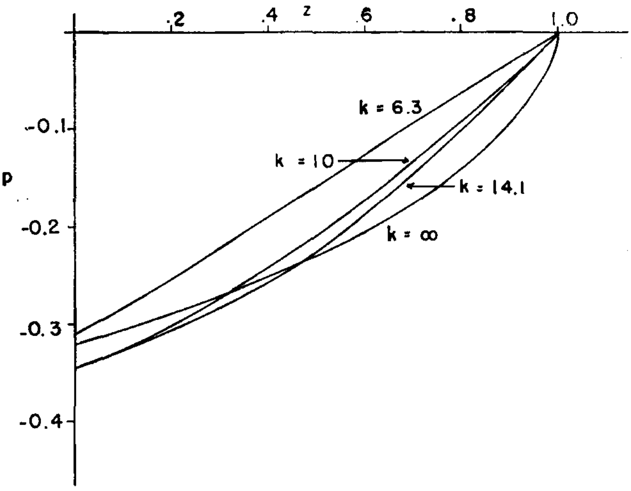

Figure 6
Dimensionless Radial Stress, Case C

If $p_0(1)$ has a finite non-zero value, the proof presented in the discussion of Case A to show that p does not have zero-th and first order boundary layer effects can be carried through. However, this has as a consequence that $p_0(1)$ must satisfy the same condition as $p(1)$, namely,

$$p_0(1) = 0, \tag{5.43}$$

which contradicts the assumption that $p_0(1) \neq 0$. This has an interesting implication; for, if $p_0(z)$ has a zero at $z = 1$ of fractional order, the above mentioned proof in Case A is no longer valid and p may possess a zero-th order boundary layer effect. This is, indeed, the way matters turn out.

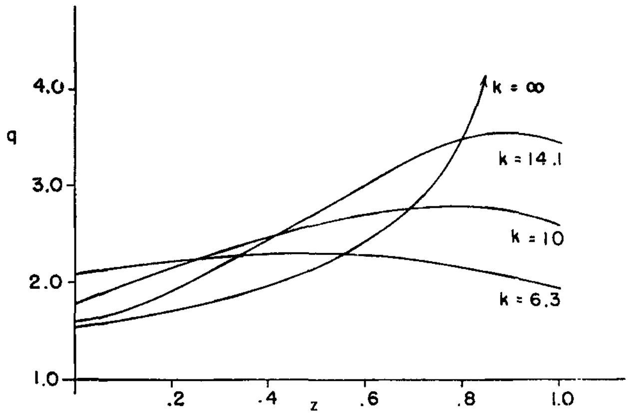

Figure 7
Dimensionless "Slope", Case C

The boundary condition (5.43) determines the value of a_0 as 1.28. The value of $q_0(z)$ can now be obtained from (5.22),

$$q_0(z) = -\frac{1}{2p_0(z)}. \tag{5.44}$$

This indicates that $q_0(z)$ has a pole at $z = 1$, and to determine its degree we reformulate the interior problem in terms of functions of x:

$$\begin{aligned} 1+2P_0Q_0 &= 0, \\ 8([1-x]P_0)''-Q_0^2 &= 0, \\ P_0(0) &= 0. \end{aligned} \tag{5.45}$$

The leading terms in P_0 and Q_0 are found to be

$$\begin{aligned} P_0(x) &= -\frac{9^{1/3}}{4}x^{2/3}, \\ Q_0(x) &= \frac{2}{9^{1/3}}x^{-2/3}, \end{aligned} \tag{5.46}$$

and these are sufficient to establish the asymptotic forms of $\bar{p}_0(y)$ and $\bar{q}_0(y)$. If, as in the introduction to this section, we set $x = \varepsilon y$, the value of ε is determined by the requirement that the boundary layer equations be non-trivial and have the same order as the original system. This yields $\varepsilon = k^{-3/4}$. The asymptotic forms of the boundary layer solutions, which also serve as the boundary conditions at $y \to \infty$, then can be written as

$$\left.\begin{aligned} \bar{p}_0(y) &\to -\frac{9^{1/3}}{4}k^{-1/2}y^{2/3}, \\ \bar{q}_0(y) &\to \frac{2}{9^{1/3}}k^{1/2}y^{-2/3}, \end{aligned}\right\} \quad \text{as } y \to \infty. \tag{5.47}$$

We now introduce the new variables

$$\begin{aligned} \tilde{p}_0(y) &= k^{1/2}\bar{p}_0(y), \\ \tilde{q}_0(y) &= k^{-1/2}\bar{q}_0(y), \end{aligned} \tag{5.48}$$

and obtain the zero-th order boundary layer equations

$$\begin{aligned} 8\tilde{q}_0''+1+2\tilde{p}_0\tilde{q}_0 &= 0, \\ 8\tilde{p}_0''-\tilde{q}_0^2 &= 0. \end{aligned} \tag{5.49}$$

Multiplying the first of these by $\tilde{q}_0'$, the second by $\tilde{p}_0'$ and subtracting the second product from the first, we obtain an expression which can be integrated to yield

$$4(\tilde{q}_0')^2-4(\tilde{p}_0')^2+\tilde{q}_0+\tilde{q}_0^2\tilde{p}_0 = 0. \tag{5.50}$$

The constant of integration has been eliminated by applying one of the conditions derived from (5.47), namely,

$$\begin{aligned}\tilde{p}_0(y) &\to -\frac{9^{1/3}}{4}\,y^{2/3},\\ \tilde{q}_0(y) &\to \frac{2}{9^{1/3}}\,y^{-2/3},\end{aligned}\qquad \text{as } y \to \infty. \tag{5.51}$$

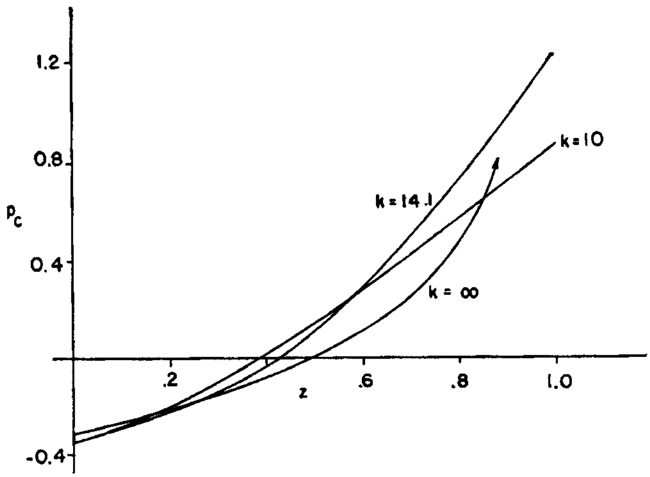

Figure 8
Dimensionless Circumferential Stress, Case C

The zero-th order boundary layer problem thus consists of the equation (5.50) and either one of the equations (5.49) together with one of the boundary conditions at infinity (5.51) and two conditions at $y = 0$, i.e.,

$$\tilde{p}_0(0) = 0, \qquad \tilde{q}_0'(0) = 0. \tag{5.52}$$

Actually, the second equation of (5.49) was chosen and a numerical integration of the problem was carried out by means of the following procedure: the system of equations can be integrated by a step-by-step method when $\tilde{q}_0(0)$ is given in addition to conditions (5.52). It can be shown that the requirement that $\tilde{q}_0$ approach zero as y approaches infinity in accordance with (5.51) implies that $\tilde{q}_0'(y) < 0$ and $\tilde{q}_0(y) > 0$ for all y. If $\tilde{q}_0(0)$ is chosen too large, it leads to positive values of $\tilde{q}_0'$ for some values of y, and if $\tilde{q}_0(0)$ is too small, it causes $\tilde{q}_0$ to become negative in the course of the integration of the system of equations. This allows us to obtain $\tilde{q}_0(0)$ to any desired accuracy by carrying out a sequence of integrations based on successively improved estimates of $\tilde{q}_0(0)$. This process also furnishes solutions for $\tilde{p}_0(y)$ and $\tilde{q}_0(y)$. Our best estimate of $\tilde{q}_0(0)$ is 0.799.

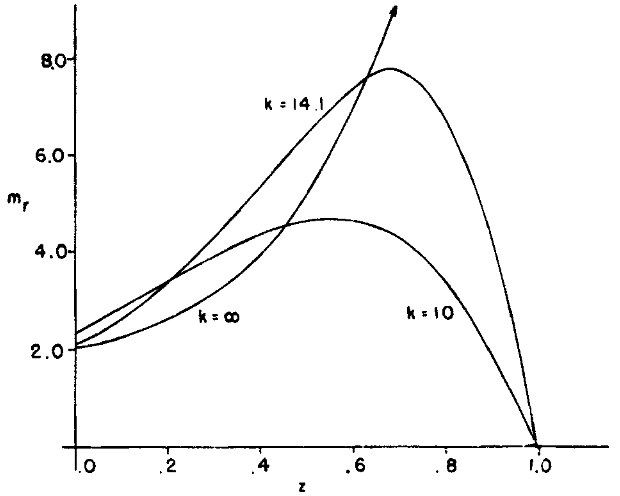

Figure 9
Dimensionless Radial Bending Moment, Case C

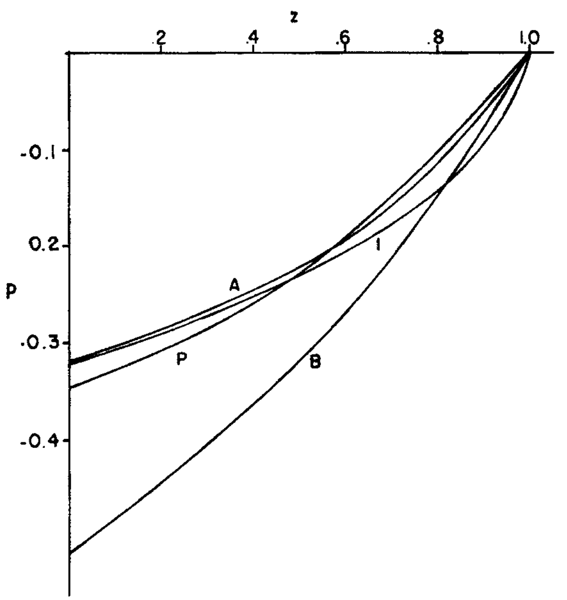

Figure 10
Dimensionless Radial Stress, Case C, $k = 14.1$
P — power series solution; $A = p_{a0}$ — approximate solution;
I — interior solution; B — boundary layer solution

As explained in the introduction to this section, the solutions $\tilde{p}_0(y)$ and $\tilde{q}_0(y)$ contain the lowest degree terms of $k^{1/2}P_0(x)$ and $k^{-1/2}Q_0(x)$. The zero-th order approximate solutions combining interior and boundary layer solutions are therefore

$$p_{a0}(z) = p_0(z) + k^{-1/2}\tilde{p}_0(y) + \frac{9^{1/3}}{4}(1-z)^{2/3},$$
(5.53)
$$q_{a0}(z) = q_0(z) + k^{1/2}\tilde{q}_0(y) - \frac{2}{9^{1/3}}(1-z)^{-2/3}.$$

These approximate solutions have been evaluated for $k = 14.1$ and the results are plotted in Figures 10 and 11, together with the power series solutions as well as the interior and boundary layer solutions, for the same value of k. It can be seen that good qualitative agreement has been achieved by means of the p_{a0} and q_{a0}.

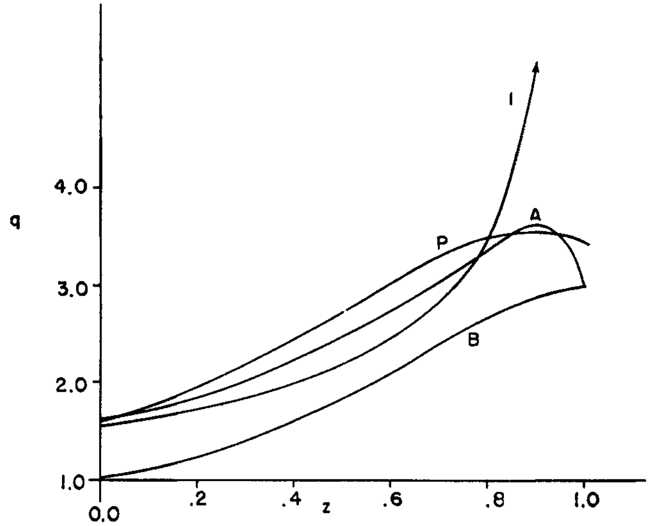

Figure 11
Dimensionless "Slope", Case C, $k = 14.1$
P — power series solution; $A = \zeta_{a0}$ — approximate solution;
I — interior solution; B — boundary layer solution

The above solutions can be used to determine the bending stresses and moments when k is large. Table II contains a summary, for the entire range of k, of the maximum stresses and moments which occur in Case C.

k^2	p_{max}	p_e (edge)	m_r max
Small	$-k^4 \times .0012$	$k^4 \times .00174$	$k^2 \times .206$
10	$-.0996$	.148	1.85
20	$-.224$	.364	2.61
40	$-.312$	.610	3.18
100	$-.345$	.867	4.85
200	$-.344$	1.238	7.45
Large	$-.321$	$k^{1/4} \times .894$	$k^{5/4} \times .216$

TABLE II

The relationship of p, p_c and m_r to the radial and circumferential stress and to the bending moment and stress are defined in Section 2.

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